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Mean first passage time in periodic attractors

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Abstract
The properties of the mean first passage time in a system characterized by multiple periodic attractors are studied. Using a transformation from a high-dimensional space to 1D, the problem is reduced to a stochastic process along the path from the fixed point attractor to a saddle point located between two neighbouring attractors. It is found that the time to switch between attractors depends on the effective size of the attractors, τ, the noise, ϵ, and the potential difference between the attractor and an adjacent saddle point as $T = \frac{\epsilon}{\tau}\exp\left(\frac{\tau}{\epsilon}\Delta\mathcal{U}\right)$; the ratio between the sizes of the two attractors affects $\Delta\mathcal{U}$. The result is obtained analytically for small τ and confirmed by numerical simulations. Possible implications that may arise from the model and results are discussed.

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1. Introduction

The problem of evaluating the mean first passage time (MFPT) has been investigated extensively in the context of chemical reactions, dynamical systems, etc [1–3]. More recently, this problem has been applied to the biological domain, e.g., kinesin walking on a microtubule [4], DNA transport through membrane channels [5] and the time to switch between protein folding states [6].

In this paper the focus is set on dynamical systems characterized by multiple periodic attractors. The system may be continuous; however, the formal derivation is given for discrete time. The problem can be viewed as follows; consider the dynamics in a phase space governed by attractors, each represents a dynamic variable. The system evolves by changing the amplitudes of these variables. The system being in a particular attractor is reflected by the dominance of the associated variable in the current system's state. In the absence of noise, the system evolves into one of the available attractors, depending on the initial condition. However, when noise is added to the dynamics the system may escape from the basin of attraction. The MFPT is then defined as the mean time it takes to drive such a system from an attractor to an adjacent saddle point (SP). A special case of this problem that arises in the context of a neural network model with a feedback loop has been treated previously. This family of model networks is in fact a system whose dynamics is influenced by the structure of the attractor space [7]. A geometrically-driven approach to the problem has been proposed in which the MFPT is calculated along a 1D path embedded in the higher dimensional attractor space. This path is a valley connecting the (periodic) fixed-point attractor and the metastable saddle point. In periodic systems and in the absence of noise, the coupled equations evolve into one of the mutually competing fixed points (FP's) at which only the Fourier components representing this attractor have a non-vanishing coefficient, i.e. only one of the periodic orbits exists asymptotically. The addition of noise generates a perturbation in each of the coupled equations. The perturbation can 'kick' the system out of the vicinity of one stable FP so that it escapes to the other FP. Our motivation is to quantify the mean time for such an event to occur.

Following the spirit of previous work the derivation is extended to arbitrary asymmetric attractors that arises from a more general nonlinear dynamical system. An analytical result for the MFPT in the limit of weak noise and a weakly nonlinear map is derived and confirmed numerically. The main reasons for taking these limits are as follows; a high level of noise may drive the system too fast from one basin of attraction to another, hence the actual dynamics of the system becomes less relevant. The reason for taking weak nonlinearity is two-fold: one is to facilitate the analytical derivation and allows for certain approximations; the other is that most of the characteristics are already revealed in this limit.

2. The model

The family of systems analysed here can be described by a set of coupled nonlinear recurrent equations, generally given by the map

$$x^i_{n+1} = f_i(X_n) + \xi_n, \tag{1}$$

where $f(X)$ is a nonlinear function, x^i is a component of X. The noise term, ξ, is a Gaussian additive noise distributed according to

$$\rho_\epsilon(\xi) = \frac{1}{(2\pi\epsilon)^{1/2}} \exp -\frac{\xi^2}{2\epsilon}. \tag{2}$$

Without loss of generality and to facilitate the derivations and the graphical presentation, we shall focus on the 2D case. In this case the map would be

$$\begin{cases} x_{n+1} = f_1(x_n, y_n) + \xi^1_n \\ y_{n+1} = f_2(x_n, y_n) + \xi^2_n. \end{cases} \tag{3}$$

Since the model deals with periodic systems, bounded nonlinear maps are of interest. In this case each attractor may be characterized by a dominant Fourier component. More specifically, the solution to the coupled equations would be some nonlinear function of the governing Fourier components. This solution would describe the time evolution of the measured variables. The assumption is that the asymptotic behaviour of the noiseless solution is a nonlinear function of the dominant Fourier components associated with that attractor. For simplicity we may assume that each attractor is associated with a single dominant component. Therefore in the following analysis we shall be interested in the amplitude of each component and one should keep in mind that in fact the underlying solution is a periodic function; however, we transform the problem to the amplitude variable. Generally speaking, the periodic function may be expanded as an infinite series; however, the demand for weak nonlinearity allows for keeping only terms up to a certain degree, e.g. third order. A typical such solution for the amplitude might be

$$A_i^{n+1} = g\left(\sum_m A_m^n \cos(\omega_m t)\right), \tag{4}$$

where $g(\,)$ is the nonlinear function and A_m is the amplitude of the 'm' Fourier component whose frequency is ω_m. The following map (5) is obtained by assuming a bounded nonlinear function that has odd terms in the polynomial expansion. Transforming the amplitude variables A_i to rescaled variables x, y and taking only the first- and third-order terms of g leads to the following map for the 2D case:

$$\begin{cases} x_{n+1} = (1+\tau_1)x_n\left[1 - ax_n^2 - by_n^2\right] \\ y_{n+1} = (1+\tau_2)y_n\left[1 - ay_n^2 - bx_n^2\right], \end{cases} \tag{5}$$

where $0 < a, b < 1$ are constants and $|x_0| < 1$, $|y_0| < 1$ (see a similar procedure in [7]).

The key point is our ability to identify a low-dimensional discrete dynamics that describes the evolution of the system, i.e., the amplitude of the dominant Fourier components in the asymptotic meta-stable state (w/o noise).

Using the following variables rescaling:

$$x = \sqrt{\tau_2}\tilde{x}, \qquad y = \sqrt{\tau_1}\tilde{y}, \tag{6}$$

gives rise to the potential function

$$\mathcal{U}(\tilde{x}, \tilde{y}) = -\frac{\tilde{x}^2 + \tilde{y}^2}{2} + \frac{a}{4}\left(\frac{\tau_2}{\tau_1}\tilde{x}^4 + \frac{\tau_1}{\tau_2}\tilde{y}^4\right) + b\frac{\tilde{x}^2\tilde{y}^2}{2}, \tag{7}$$

where the coupled equations are obtained via

$$\tilde{x}_{n+1} = \tilde{x}_n - \tau_1\mathcal{U}'_x, \qquad \tilde{y}_{n+1} = \tilde{y}_n - \tau_2\mathcal{U}'_y.$$

Keeping only terms with the small parameters $\tau_{1,2}$ to first order, the following transformed equations are obtained:

$$\begin{cases} \tilde{x}_{n+1} = \tilde{x}_n - \tau_1\left(-\tilde{x}_n + a\frac{\tau_2}{\tau_1}\tilde{x}_n^3 + b\tilde{x}_n\tilde{y}_n^2\right) \\ \tilde{y}_{n+1} = \tilde{y}_n - \tau_2\left(-\tilde{y}_n + a\frac{\tau_1}{\tau_2}\tilde{y}_n^3 + b\tilde{y}_n\tilde{x}_n^2\right). \end{cases} \tag{8}$$

The fixed points of the dynamics (assuming $\tau_i \ll 1$) are given (for the rescaled variables) by

$$\left[\pm\sqrt{\frac{\tau_1}{a\tau_2}}, 0\right], \qquad \left[0, \pm\sqrt{\frac{\tau_2}{a\tau_1}}\right], \qquad \left[\pm\sqrt{\frac{a}{a^2-b^2}\left(\frac{\tau_1}{\tau_2} - \frac{b}{a}\right)}, \pm\sqrt{\frac{a}{a^2-b^2}\left(\frac{\tau_2}{\tau_1} - \frac{b}{a}\right)}\right], \tag{9}$$

where the last (four) FP are in fact SPs and the first (four) are stable FP. An example of the phase portrait in this case is given in figure 1.

To visualize the potential surface an example is depicted in figure 2; for clarity the negative potential is shown—darker area means more negative potential (e.g., the zero FP is not stable). The variables shown are rescaled and the actual values of the potential are not important. It is apparent that there exists a path from a non-zero FP to adjacent SP via a valley (here a hill).

As mentioned above, an important step in the derivation is the conjecture that the most probable trajectory from a FP to one of its nearest SP defines the properties of the MFPT. In the next section the properties of the rescaled 1D noisy map with a potential given by equation (7) are derived. By fixing the initial and final points the path is unique; hence,

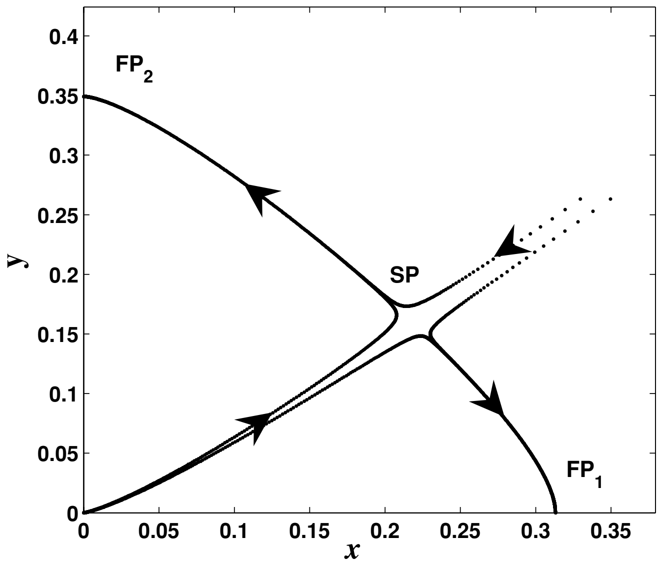

Figure 1. Phase portrait of a 2D map with $\tau_1 = 0.02$, $\tau_2 = 0.025$, $a = 0.2$, $b = 0.4$. 'SP' denotes a saddle point and 'FP' a fixed point. Arrows show the direction of the flow.

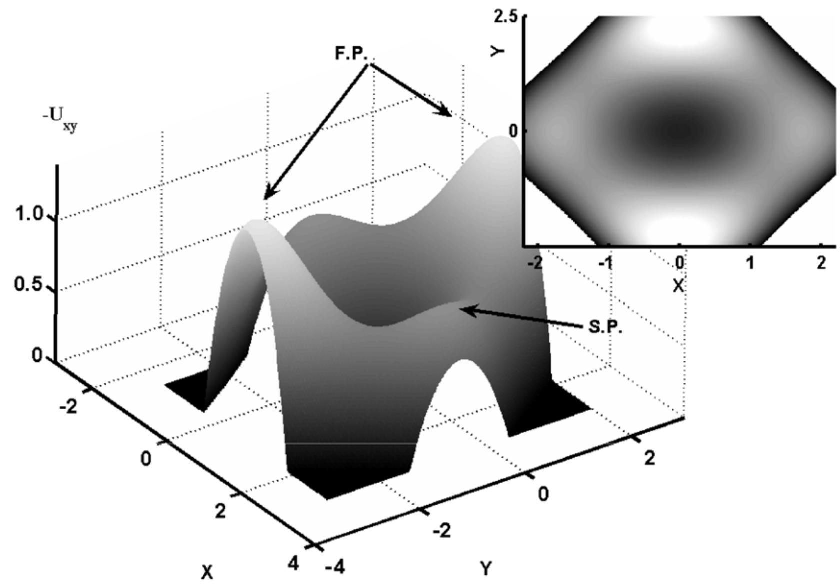

Figure 2. Negative potential surface of the function given in equation (7). The inset is the contour plot. The parameters used were $\frac{\tau_1}{\tau_2} = 3/4$, $a = 0.5$, $b = 0.25$. 'SP' denotes a saddle point and 'FP' a fixed point.

the projection of near-by trajectories on the path is treated and the noise term is rescaled appropriately.

It should be emphasized again that the variables in the family of maps we study represent the amplitude of the governing Fourier components in the asymptotic solution, so a fixed point in this map is analogous to a stable periodic solution in the original system.

3. Escape from a meta-stable attractor

To illustrate the idea behind the following derivation let us assume a scenario where the initial condition is $y = 0, x = x^\star$, i.e. one of the FP's of the dynamics (refer to figure 1). Since the line connecting this FP and the SP is a valley (in the potential space), it may be assumed that the most probable escape route is along this line (or its mirror through the x-axis, i.e. the line connecting the FP with the SP $(x^+_{\mathrm{SP}}, y^-_{\mathrm{SP}})$). This argument can be understood by rotating each noise term tangent and perpendicular to the path. The perpendicular term decays fast due to the restoring force; hence, the crucial step that the dynamics is mainly 1D is conjectured. It should be noted that this conjecture is not limited to our 2D example; rather, a set of attractors embedded in a higher dimensional space should exhibit the same property, namely, that the most probable escape route from the domain of attraction is through the valley connecting this attractor and a SP residing between two adjacent FP's. Therefore, with the assumption of weak noise and $\tau \ll 1$ the map can be reduced to one dimension, on that path; hence, a 1D noisy map is obtained

$$s_{n+1} = s_n - \tau \mathcal{U}'(s_n) + \hat{\xi}_n, \tag{10}$$

where s defines the path. The noise term now is the tangential projection of the noise vector on the path. The explicit form of the path is not crucial as long as smoothness and monotonicity are guaranteed; since it is assumed that there are no extremums between the FP and the SP this assumption holds.

This type of a 1D equation has been investigated by several authors [2, 8, 9] for the case of small nonlinearity, namely, the class of map functions with the property that $f(s)$ deviates only weakly from the identity map:

$$f(s) = s - \tau \frac{\mathrm{d}U(s)}{\mathrm{d}s}, \quad \tau \ll 1 \tag{11}$$

in analogy with our 1D map (equation (10)) .

Assume that the process described in equation (10) is defined in $(-\infty, \infty)$ and let us define the random variable $\tilde{t}(s)$, the first passage time from the interval $I = [\mathrm{SP}^-, \mathrm{SP}^+]$, by

$$\tilde{t} = \min\left\{n : |s_n| \geqslant s^+_{\mathrm{SP}}\right\}, \tag{12}$$

i.e. the first time the process hits one of the boundaries, where $\mathrm{SP}^\pm$ are the saddle points defined above and s^+_{SP} is the value of s at the saddle point. The MFPT, $t(s)$, starting from a point in I is given by

$$t(s) = \langle \tilde{t}(s) \rangle = E[\tilde{t}|S_0 = s]. \tag{13}$$

It was shown (e.g., [9]) that the MFPT can be written as

$$t(s) - 1 = \int_I P(z|s) t(z)\, \mathrm{d}z, \tag{14}$$

where $P(z|s)$ denotes the transition probability to go from $s_n = s$ to $s_{n+1} = z$ in a single step.

In the next subsections the probability density function and the MFPT are derived.

3.1. Probability density function

Assume the noise terms, ξ_n, are Gaussian distributed and mutual independent; hence, the stochastic process s_n is a Markov process. The distribution is given by equation (2), with a rescaled amplitude ϵ. The transition probability from $s_n = z$ to $s_{n+1} = s$ in one step is

$$P(s|z) = \rho(s - f(z)), \tag{15}$$

with $\rho(\xi)$ given by equation (2). The probability density, $Q_n(s)$, to find the system in s after n steps evolves according to

$$Q_{n+1}(s) = \int_{-\infty}^{\infty} P(s|z) Q_n(z)\, \mathrm{d}z. \tag{16}$$

In a similar way, one can define the conditional probability $P_n(s|z)$ to get from z to $[s, s + \mathrm{d}s]$ in n steps. The invariant probability density $Q(s)$ is the solution of equation (16). Plugging equation (15), the invariant density obeys

$$Q(s) = \int_{-\infty}^{\infty} P(s|z) Q(z)\, \mathrm{d}z = (2\pi\epsilon)^{-1/2} \int \exp\left\{-\frac{1}{2\epsilon}[s - z + \tau\mathcal{U}'(z)]^2\right\} Q(z)\, \mathrm{d}z. \tag{17}$$

This integral cannot be solved in general. In the case of weak nonlinearity, e.g. our map with $\tau \ll 1$, one can solve the integral approximately by setting $u = (2\epsilon)^{-1/2}(s - z + \tau\mathcal{U}'(z))$ and neglecting $\mathcal{O}(\tau^2)$ terms in the expansion of $z(u)$. Alternatively, one can assume a WKB-type solution (e.g., [10]) to solve equation (17) for weak noise, $\epsilon \ll 1$,

$$Q(s) = N(s) \exp\left(-\frac{\Phi(s)}{\epsilon}\right), \tag{18}$$

where $\Phi(s)$ is a potential function and $N(s)$ is a normalization factor. Inserting this type of solution back in equation (17), one obtains

$$N(s) = (2\pi\epsilon)^{-1/2} \int N(z) \exp\left[-\frac{1}{2\epsilon} W^2(s, z)\right] \mathrm{d}z, \tag{19}$$

where

$$W^2(s, z) = 2[\Phi(z) - \Phi(s)] + [s - f(z)]^2, \tag{20}$$

whereas in our model $f(z) = z - \tau\mathcal{U}'(z)$. The demand for (semi) positivity of W^2 implies that the potential $\Phi(s)$ is a Lyapunov function of the noise-free dynamics. This is evident by setting $s = f(z)$ in equation (20). To obtain an equation for the potential $\Phi(s)$ one can derive a transformation from the old variable, z, to the new one, W. It is sufficient and necessary that $\partial W(s, z)/\partial z \neq 0$ to write

$$N(s) = (2\pi\epsilon)^{-1/2} \int N(z(W, s))[\partial W(s, z)/\partial z|_{z=z(W,s)}]^{-1} \exp\left[-\frac{1}{2\epsilon} W^2\right] \mathrm{d}W, \tag{21}$$

where $z(W, s)$ is a solution of equation (20). Assuming $N(z(y, s))[\partial W/\partial z|_{z=z(W,s)}]^{-1}$ is smooth in a neighbourhood of $\mathcal{O}(\epsilon^{1/2})$ around $W = 0$, the integral can be approximated for weak noise by

$$N(s) = N(z(y = 0, s))[\partial W/\partial z|_{z=z(W=0,s)}]^{-1}. \tag{22}$$

Assuming the variable transformation remains valid ($\partial W/\partial z \neq 0$) for the case $\partial W^2(s, z)/\partial z = 0$, W^2 must vanish as well. Therefore, differentiating (20) leads to an equation for s. Plugging $W^2 = 0$ and the equation for s back to equation (20), the desired relation for $\Phi(s)$ is obtained:

$$\Phi(z) - \Phi\left(\frac{\Phi'(z)}{1 - \tau\mathcal{U}''(z)} + z - \tau\mathcal{U}'(z)\right) + \frac{1}{2}\left(\frac{\Phi'(z)}{1 - \tau\mathcal{U}''(z)}\right)^2 = 0, \tag{23}$$

where the map f has been written explicitly and $\mathcal{U}''(z)$ is a second-order derivative w.r.t. z. The solution of this equation for small nonlinearity, $\tau \ll 1$, gives finally

$$Q(s) = N \exp\frac{-2\tau\mathcal{U}(s)}{\epsilon}, \tag{24}$$

where the prefactor N is constant up to corrections of $\mathcal{O}(\tau^2)$.

3.2. The MFPT

Turning now to the analysis of the MFPT, consider the random variable $\tilde{t}(s)$—the random time to leave the domain of attraction, I, for the first time, starting from a point $s \in I$. The probability that after n steps the process does not leave the domain I is given by

$$Q(n|s) = \int_I P_n(z|s)\,\mathrm{d}z, \tag{25}$$

where $P_n(z|s)$ is the probability to get from s to z in exactly n steps, obeying a similar equation to (16)

$$P_{n+1}(z|s) = \int_I P_n(z|y)P_1(y|s)\,\mathrm{d}y, \tag{26}$$

where this recursive relation uses the one-step probability P_1. Since $Q(n|s)$ is a decreasing function of the discrete time n, the probability that an exit occurs exactly after n steps is simply $Q(n|s) - Q(n+1|s)$; hence, the first moment of our random variable, $\tilde{t}(s)$, is

$$t(s) = \langle \tilde{t}(s)\rangle = \sum_{n=0}^{\infty}(n+1)(Q(n|s) - Q(n+1|s)) = \sum_{n=0}^{\infty} Q(n|s). \tag{27}$$

Summing equation (16) over n and taking the integral inside the domain I only, one obtains

$$\sum_{n=0}^{\infty} Q(n+1|s) = \int_I \sum_{n=0}^{\infty} Q(n|s)P_1(z|s)\,\mathrm{d}z. \tag{28}$$

On noting that the lhs equals $t(s) - 1$ $(Q(0|s) = 1\ \forall\ s \in I)$ this equation becomes equation (14). Multiplying equation (14) by the invariant probability density $Q(s)$ and integrating over the domain as described by equation (12), i.e., $I = [\mathrm{SP}^-, \mathrm{SP}^+]$, it is found that

$$\int_{\mathrm{SP}^-}^{\mathrm{SP}^+} Q(s)\,\mathrm{d}s = \left(\int_{-\infty}^{\mathrm{SP}^-}\mathrm{d}s + \int_{\mathrm{SP}^+}^{\infty}\mathrm{d}s\right) Q(s)\int_{\mathrm{SP}^-}^{\mathrm{SP}^+} P_1(z|s)t(z)\,\mathrm{d}z. \tag{29}$$

Under the assumption of weak noise $\epsilon \ll 1$, the function $t(s)$ is nearly constant inside the domain of attraction. Fluctuations occur mainly near the boundary. The reason is that only close to the boundary may one have a finite probability to jump over the boundary in a small number of steps. Therefore, it was suggested (see [8, 9]) that this function can be written as a product of a constant value and a boundary layer function

$$t(s) = T\tilde{h}(s), \qquad \tilde{h}(s^\star) = 1, \tag{30}$$

where $s^\star$ is the FP. The boundary layer extends a distance of order $\epsilon^{1/2}$ around $s = s^{\pm}_{\mathrm{SP}}$, and the scaled boundary layer function $h(s)$ may be written as $h(s) = \tilde{h}((2\epsilon)^{1/2}s)$.

Using the boundary layer function $\tilde{h}$ (equation (30)) to account for the fact that for weak noise $t(s)$ is almost constant inside the domain, except near the boundaries, MFPT (starting from the FP) is given by

$$T^{-1} = \frac{\left(\int_{-\infty}^{\mathrm{SP}^-}\mathrm{d}s + \int_{\mathrm{SP}^+}^{\infty}\mathrm{d}s\right)Q(s)\int_{\mathrm{SP}^-}^{\mathrm{SP}^+} P_1(z|s)\tilde{h}(z)\,\mathrm{d}z}{\int_{\mathrm{SP}^-}^{\mathrm{SP}^+} Q(s)\,\mathrm{d}s}, \tag{31}$$

where the first integral in the numerator is the inverse MFPT from the negative exit and the second from the positive. Since the SPs are symmetric w.r.t. the FP the integrals are equal. The invariant density is peaked around the stable FP; hence, the steepest descent approximation may be used for both the numerator and the denominator of equation (31). Expanding the map near the FP,

$$f(z^\star + \Delta z) \approx z^\star + \Delta z f'(z^\star) = z^\star + \Delta z(1 - \tau \mathcal{U}''(z^\star)), \tag{32}$$

where $z^\star$ denotes the FP, and using the Gaussian approximation of equation (17), the denominator of equation (31) becomes

$$\int_{\mathrm{SP}^-}^{\mathrm{SP}^+} Q(s)\,\mathrm{d}s \approx Q(\mathrm{FP}) \left[\frac{\pi\epsilon}{\tau \mathcal{U}''(\mathrm{FP})}\right]^{1/2}, \tag{33}$$

where the weak nonlinearity assumption, $\tau \ll 1$, is used. The numerator is evaluated in a similar manner yielding

$$\left(\int_{-\infty}^{\mathrm{SP}^-} \mathrm{d}s + \int_{\mathrm{SP}^+}^{\infty} \mathrm{d}s\right) Q(s) \int_{\mathrm{SP}^-}^{\mathrm{SP}^+} P_1(z|s)\tilde{h}(z)\,\mathrm{d}z \approx \epsilon^{1/2} G(\tau) Q(\mathrm{SP}), \tag{34}$$

with $G(\tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} + O(\tau)$.

Combining the terms, the main result is obtained:

$$T = \frac{\mathcal{C}}{\tau}\frac{Q(\mathrm{SP})}{Q(\mathrm{FP})} = \frac{\mathcal{C}}{\tau} \exp \frac{2\tau}{\epsilon}\left(\mathcal{U}(\mathrm{SP}) - \mathcal{U}(\mathrm{FP})\right), \tag{35}$$

where $\mathcal{C}$ is a constant. Note that the time T is rescaled w.r.t. a cycle of the attractor.

Plugging the general form of the FP's of the dynamics (equation (9)) in equation (7), an explicit expression for the potential terms in equation (35) as a function of the variables of the model, τ_1, τ_2, a, b, is obtained as follows:

$$\mathcal{U}(\mathrm{SP}) = -\frac{a\tau_{12}^2 + a - 2b\tau_{12}}{4\tau_{12}(a^2 - b^2)} \qquad \mathcal{U}(\mathrm{FP}) = -\frac{\tau_{12}}{4a}\left(-\frac{1}{4a\tau_{12}}\right) \tag{36}$$

where $\tau_{12} = \tau_1/\tau_2$; the parentheses on the rhs are given for the second FP. Note that τ_1, τ_2 are given now w.r.t. a common variable τ, e.g., $\tau_2 = \tau$ and $\tau_1 = \tau_{12}\tau$. The special case $\tau_1 = \tau_2 = \tau$ reduces to $(\mathcal{U}(\mathrm{SP}) - \mathcal{U}(\mathrm{FP})) = (b - a)/(4a(a + b))$, and for the choice $a = 1/4, b = 1/2$ used below for simulations, $\Delta\mathcal{U} = 1/3$.

3.3. Numerical simulations

To confirm the theoretical results extensive numerical simulations of the model were performed. Equations (8) were used to evaluate the statistical properties of the MFPT. For each trial, a line that passes through the SP and perpendicular to the line connecting the FP and the SP was constructed. The time to hit this line starting from the FP was collected in 500–1000 trials for each choice of the parameters. The model parameters used in the figure were $a = 1/4, b = 1/2$ (although other values were tested as well). Typically the attractor and noise parameters were chosen such that $\tau/\epsilon \in (2 \ldots 7)$, and $0.8 \leqslant \tau_{12} \leqslant 1.2$.

To demonstrate the scaling properties of the results, equations (35)–(36), the logarithm of the MFPT is evaluated and the following quantity is depicted in the next figure:

$$\langle \ln T\rangle + \ln\tau \propto \tau/\epsilon.$$

The slope of each series of simulations corresponds to a specific choice of $\tau_{12}, \tau, \epsilon$. The simulation results shown in figure 3 are in a good agreement with the predicted values calculated using equation (36), see table 1.

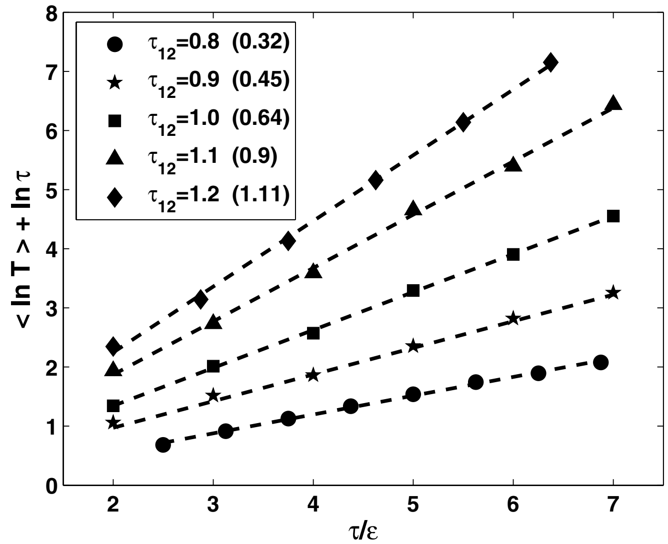

Figure 3. Scaling of the average logarithm of the escape time in the 2D model. The dashed lines are the linear regression of the corresponding data points. The standard deviation of each point is roughly the same as the symbol, hence omitted for clarity. The legend shows for each $\tau_{12} = \tau_1/\tau_2$ the estimated slope in parentheses, see also table 1.

Table 1. Predicted versus simulated potential difference values.

τ_{12}	$2(\mathcal{U}(\mathrm{SP}) - \mathcal{U}(\mathrm{FP}))$	Simulation
0.8	0.3	0.32 ± 0.02
0.9	0.47	0.45 ± 0.03
1.0	0.66	0.64 ± 0.03
1.1	0.88	0.90 ± 0.03
1.2	1.09	1.11 ± 0.03

4. Discussion

The theory for the mean first passage time to escape a periodic attractor defined by the particular fixed point of the dynamic equations was developed. To facilitate the analytical investigation, proper variables transformations were identified that decompose the potential function of the system. The reduced variables are closely related to the amplitude of the solution. The noiseless system relaxes to one of the stable non-zero solutions (above bifurcation). Adding noise to the dynamics perturbs this solution and enables possible escape.

One of the key points in the solution is the conjecture that the most probable escape route is essentially one dimensional along the path from the attractor to one of the adjacent saddle points. This enabled us to reduce the dimensionality of the dynamics into a 1D process. Taking the limit of small noise and not too far from the bifurcation the theory developed for noise-driven discrete dynamical systems has been applied. The results resemble those obtained in systems with the potential barrier undergoing a tunnelling in the sense that the escape time has a polynomial prefactor and a leading exponential term.

Simulations of the system with two variables have shown that the theory developed, and especially the reduction to a 1D flow, are applicable and provide a good prediction of the exponential term, as well as the polynomial prefactor.

It should be noted that the results are applicable to non-periodic systems as well, as long as one can transform the dynamic equations to a form similar to equations (5), i.e., up to third order. Also, systems with higher order terms may exhibit a very similar behaviour assuming these terms do not dominate the solutions and the discussion is restricted to solutions not too far from bifurcation, i.e. $\tau \ll 1$.

The implications that arise from these results may have practical application in fields such as optimization, minimization problems, numerical analysis, parameter estimation in high-dimensional complex systems (e.g., neural networks) and more. The common ground to these problems is the existence of (usually) high-dimensional attractor space in which some dynamics is imposed, e.g., obtaining a minimal energy solution to optimization problem via some iterative algorithm. This algorithm evolves the solution preferably to a global minimum; however, it may get trapped in local minima. To escape these attractors, one occasionally perturbs the solution to escape from the solution etc. It is of practical importance to assess the time to escape (affect simulation time) and possibly the required amplitude and duration of perturbation. All these issues are left for future research. In this context it is interesting to mention a work by Baronchelli and Loreto [11] on mean first passage time in graphs, where the focus is on the complexity of evaluating the MFPT on a particular node of a graph.

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